Generalized determinant solution of the discrete-time totally asymmetric exclusion process and zero-range process

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We consider the discrete-time evolution of a finite number of particles obeying the totally asymmetric exclusion process with backward-ordered update on an infinite chain. Our first result is a determinant expression for the conditional probability of finding the particles at given initial and final positions, provided that they start and finish simultaneously. The expression has the same form as the one obtained by Schütz [J. Stat. Phys. **88**, 427 (1997)] for the continuous-time process. Next we prove that under some sufficient conditions the determinant expression can be generalized to the case when the particles start and finish at their own times. The latter result is used to solve a nonstationary zero-range process on a finite chain with open boundaries.

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I. INTRODUCTION

The one-dimensional asymmetric simple-exclusion process (ASEP) has been intensively studied over decades by both physicists [1–3], and mathematicians [4,5]. The model can be used to describe such different physical problems, including kinetics of biopolymerization [1], traffic flow [6], surface growth [7,8], and shock structures [9]. For a more complete list of works we refer the reader to the reviews [10,11].

The totally asymmetric version (TASEP) is one of the simplest examples of driven lattice-gas systems with hard-core exclusion. Most often, the evolution of the system is considered as a continuous-time stochastic process in which particles jump randomly and independently at a unit rate to a neighboring vacant site on the right. In the case of a finite chain with open boundaries, particles are injected and removed with specified rates at the ends. The probabilistic cellular automaton analog of the continuous-time ASEP is the discrete-time ASEP, which is defined by update rules of the system configurations (for a description of the basic update rules, see [12]).

By now, the steady state properties of the continuous- and discrete-time ASEP are well understood and some of them have been calculated exactly for both infinite and finite chains under different boundary conditions. The matrix-product ansatz (MPA) has been successfully applied for constructing the stationary states of the TASEP with open boundaries for all basic types of stochastic dynamics: random-sequential [13], forward- and backward-ordered sequential [14], sublattice parallel [15], and fully parallel update [16,17]. One of the most important findings is the existence of boundary induced phase transitions between steady states driven out of equilibrium by nonvanishing currents of particles. Other interesting phenomena concern the time-dependent properties of the ASEP, e.g., the strong dependence of the fluctuations on the initial conditions (see [9] and references therein). In contrast, almost all the dependence on the initial conditions is eliminated in the stationary states.

The structure of transient states is more complicated and the description of their time evolution is much harder to obtain [18,19]. The MPA approach, extensively used for the description of the stationary states of ASEP, has been generalized to the full dynamic problem by Stinchcombe and Schütz [20,21]. Later, a new type of dynamic MPA, which differs from the former in the time dependence of the matrices, has been suggested by Sasamoto and Wadati [22].

A different approach to the time-dependent properties of the ASEP has been proposed by Schütz [23]. It is based on the explicit solution of the master equation for the conditional probability $P^{(n)}(x_1,\dots,x_n;t|x_1^0,\dots,x_n^0;0)$ of finding a finite number n of particles on lattice sites $x_1,\dots,x_n$ at time t, provided that initially they have occupied the set of sites $x_1^0,\dots,x_n^0$. In the case of an infinite chain the solution has been obtained in the form of determinant of a $n\times n$ matrix.

Among a variety of interpretations of the TASEP, the formulation of the process in terms of traffic flow in discrete time is one of the most transparent (see, e.g. [24]). Despite the spatial discretization, the exclusion interaction between particles mimics the motion of cars on a single lane. On the other hand, the traffic analogy supplies the theory of the TASEP with some new problems. Firstly, traffic is most adequately represented by the stochastic discrete-time parallel update, not to mention the need for more sophisticated update rules [6,24]. Our first aim in this paper is to show that the conditional probability $P^{(n)}(x_1,\dots,x_n;t|x_1^0,\dots,x_n^0;0)$ for the *discrete-time* TASEP with backward-ordered update [12] on an infinite chain can be obtained in a determinant form quite similar to that found by Schütz [23].

Secondly, realistic traffic takes place on roads with local inhomogeneities such as road crossings, on- and off-ramps, changing number of lanes, etc. Recent investigations have focused on the various nonequilibrium phases of congested

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traffic (e.g., localized clusters, stop-and-go waves, different kinds of "synchronized" traffic) caused by inhomogeneities in the bulk. The existence of such phases has been predicted by appropriate cellular automaton models (see [25–27]), and some of them have been empirically observed [27,28]. Here, we emphasize that the appearance and disappearance of cars at given sites of the road, at given moments of time, correspond in the TASEP to a generalized (unequal-time) conditional probability $P^{(n)}(x_1,t_1;\ldots;x_n,t_n|x_1^0,t_1^0;\ldots;x_n^0,t_n^0)$ with different pairs of discrete space-time coordinates (x_i^0,t_i^0) and (x_i,t_i) of creation and annihilation, respectively, of the different particles.

It is the second goal of this paper to present an exact expression for the unequal-time conditional probability $P^{(n)}(x_1,t_1;\ldots;x_n,t_n|x_1^0,t_1^0;\ldots;x_n^0,t_n^0)$ similar to the determinant formula obtained in [23]. In principle, the domain of validity of the determinant formula derived here could be obtained by dissecting the relevant time interval into subintervals with a fixed number of particles k and then applying the equal-time probabilities $P^{(k)}(x_1,\ldots,x_k;t|x_1^0,\ldots,x_k^0;0)$ to each subinterval. This procedure implies, however, intermediate summations over the coordinates of all the particles on the common boundaries of the adjacent subintervals; hence, it seems rather cumbersome and cannot be realized for arbitrary arrangements of the starting and ending points of the trajectories. We can prove our new result for the TASEP with backward-ordered update under some *sufficient* conditions on the space-time endpoints of the particle trajectories.

The unequal-time probability allows one to consider a number of kinetic problems with a variable number of particles entering and leaving the system. The most interesting problem of such a kind is the TASEP on a finite chain, with prescribed moments of time $t_1^0,t_2^0,\ldots,t_n^0$ at which the particles enter the system at its left end, and prescribed moments of time $t_1,t_2,\ldots,t_n$ at which they leave the system at its right end. As it is seen from the conditions of Theorem 2, this case is out of the range of validity of the proof of the determinant formula. Nevertheless, there is an important stochastic process, the so-called zero-range process (ZRP) [4], which can be considered on a finite chain and solved by mapping on a TASEP problem with known unequal-time conditional probability. It is the third goal of our paper to consider time-dependent properties of such a discrete-time ZRP. The ZRP is one of the basic models of queueing theory; it is also widely used for the description of sandpile dynamics [29], drop-push dynamics of a fluid in a porous medium [30], surface growth phenomena, etc. The unequal-time probabilities for the ZRP open new prospects for the exact evaluation of various time-dependent correlations which cannot be obtained by the existing methods.

The continuous-time results follow from our expressions by taking a straightforward limit, which amounts to the substitution of the Bernoulli distribution by the Poisson one.

The structure of the paper is as follows. In Sec. II we define the discrete-time TASEP with backward-ordered dynamics for a finite number of particles and prove Theorem 1 which yields an extension of the determinant formula [23] for the equal-time conditional probabilities. In Sec. III we further generalize the consideration to the case of different times of injection and removal of each particle. The main result of this section, Theorem 2, establishes the existence of a generalized determinant formula under some sufficient conditions. The utility of our results is illustrated by an application to the study of the stochastic dynamics of the ZRP on a finite chain in Sec. IV. The paper closes with Sec. V, where we discuss the difference between determinant expressions enumerating mutually, avoiding trajectories in the class of free-fermion models and in the ASEP models studied here.

II. THE DISCRETE-TIME TASEP ON INFINITE CHAIN

In this section, instead of explicitly solving the discrete-time master equation, we make use of the geometrical treatment of the Bethe ansatz developed in [31] to analyze entangled systems of allowed and forbidden trajectories of interacting particles on the infinite chain.

The discrete space-time version of the TASEP is defined as follows. Consider the infinite triangular lattice Λ obtained from the square lattice by adding a diagonal between the upper left and lower right corners of each elementary square. Let (x,t) be the integer space-time coordinates of a particle on Λ, where the vertical time axis is directed down and the horizontal space axis is directed to the right. A trajectory of a particle is a sequence of connected vertical and diagonal bonds of Λ. Each diagonal bond corresponds to a jump of the particle to its nearest neighbor on the right for unit time and has a statistical weight z. Each vertical bond corresponds to a stay of the particle at the site corresponding to its spatial coordinate during the unit time interval and has a statistical weight y. The statistical weight of all the one-particle trajectories starting at the point (x^0,t^0) and ending at the point (x,t) is

$$B_0(N;T) = \binom{T}{N} z^N y^{T-N}, \tag{1}$$

where $N=x-x^0$ is the distance traveled for time $T=t-t^0$. Obviously, for the totally asymmetric process, $B_0(N;T)\neq 0$ if and only if $0\leqslant N\leqslant T$. To provide $B_0(N;T)$ with probabilistic meaning, we put $0<z<1$ for the probability of one spatial step to the right, hence $y=1-z$ becomes the probability of an isolated particle to stay at the same site during unit time interval. Then $B_0(N,T)$ is the probability to reach the point (N,T) from the origin $(0,0)$. To obtain the continuous-time limit, we set $T=Mt$, $z=1/M$, and pass to the limit $M\to\infty$,

$$\lim_{M\to\infty} B_0(N;Mt)\big|_{z=1/M} = \frac{t^N}{N!}e^{-t} := F_0(N;t), \tag{2}$$

where t is the rescaled continuous time. Thus, the above limit leads to replacement of the Bernoulli distribution by its Poisson analog.

Next we formulate the standard n-particle problem for the discrete space-time TASEP. Consider the set of trajectories of n particles on the lattice Λ which start at the points $(x_1^0,0),\ldots,(x_n^0,0)$, $x_1^0<x_2^0<\ldots<x_n^0$ and end at the points $(x_1,t),\ldots,(x_n,t)$, $x_1<x_2<\ldots<x_n$. The exclusion rules read as follows:

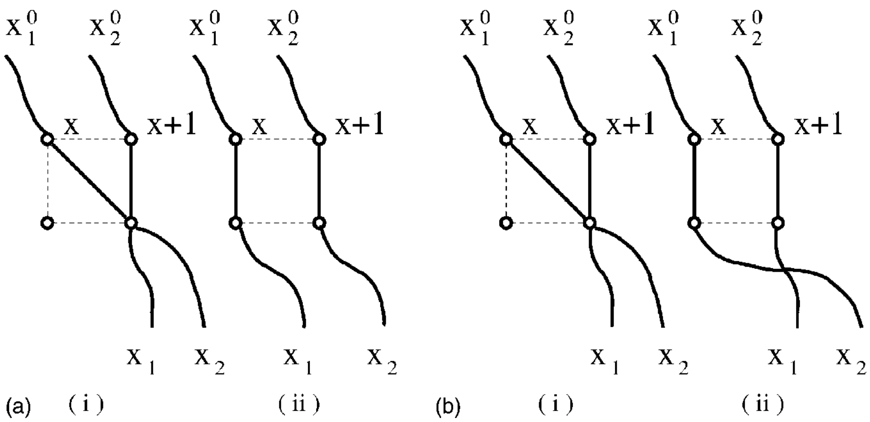

FIG. 1. The interaction between two trajectories (see text).

(a) Trajectories of particles do not intersect.

(b) If the two vertical bonds of any elementary square of Λ are occupied by adjacent trajectories, the weight of the left bond is changed from $1-z$ to 1.

Rule (a) is the usual condition for occupation of every site by at most one particle. Rule (b) implies that the particle stays at the given site with probability 1 if the target site is occupied by a standing particle.

The equal-time conditional probability $P_n(x_1,\ldots,x_n;t|x_1^0,\ldots,x_n^0;0)$ of the discrete TASEP is given by the weighted sum over all trajectories of n particles, starting from the given set of sites at time 0 and ending at the given set of sites at time t, which are allowed by the exclusion rule (a) and have weights corrected according to rule (b).

To formulate our first result, we define the discrete-time analogs of the functions introduced in [23]:

$$B_m(N;T) = \sum_{k=0}^{\infty} \binom{k+m-1}{m-1} B_0(N+k;T) \tag{3}$$

for integer $m>0$, and

$$B_m(N;T) = \sum_{k=0}^{-m} (-1)^k \binom{-m}{k} B_0(N+k;T) \tag{4}$$

for integer $m<0$; for $m=0$, $B_0(N,T)$ is given by Eq. (1).

The derivation of this result is based on a common property of integrable models admitting a two-dimensional graphic representation: interchanging the endpoints of two trajectories leads to their crossing. The idea of the Bethe ansatz is to represent trajectories of interacting particles by a set of free trajectories with probabilities given by Eq. (1) or Eq. (2). Then, using the one-to-one correspondence between intersections and permutations, one can reduce the enumeration of all the interacting trajectories to a proper choice of the signs of permutations.

Let us start with the case of two particles, $n=2$. According to the Bethe ansatz, we try to represent the motion of interacting particles by free trajectories from $(x_i^0,0)$ to (x_i,t), $i=1,2$. Consider an elementary square of Λ with space coordinate x of the left-hand side and $x+1$ of the right-hand side. Assume that the particles come for the first time to neighboring sites at a moment t', when one trajectory reaches the site (x,t') from $(x_1^0,0)$ and the other reaches the site $(x+1,t')$ from $(x_2^0,0)$. To ensure the correct weights of the steps after the moment of time t', we have to exclude two possibilities from all continuations of the interacting trajectories [see Fig. 1(a)], namely:

(i) For the first particle, the step from (x,t') to $(x+1,t'+1)$ with weight z, and then from $(x+1,t'+1)$ to (x_1,t). For the second particle, the step from $(x+1,t')$ to $(x+1,t'+1)$ with weight $y=1-z$, and then from $(x+1,t'+1)$ to (x_2,t).

(ii) For the first particle, the step from (x,t') to $(x,t'+1)$ with weight $y-1=-z$, and then from $(x,t'+1)$ to (x_1,t). For the second particle, the step from $(x+1,t')$ to $(x+1,t'+1)$ with weight $y=1-z$, and then from $(x+1,t'+1)$ to (x_2,t).

Case (i) is the forbidden step of the first particle toward the site of the standing second particle. Case (ii) is a correction of the weight of the vertical step of the first particle, which must be 1 instead of $y=1-z$, according to the TASEP rule (b). The generating function of paths of the first particle in case (i) is a product of three factors: $B_0(x-x_1^0,t')zB_0(x_1-x-1,t-t'-1)$. Let $W(a,x|z|x+1,b)$ be the generating function of all the one-particle trajectories passing through the sites $a,x,x+1,b$ at moments of time $0,t',t'+1,t$, respectively, and making a diagonal step with weight z between t' and $t'+1$. Similarly, let $W(a,x|y|x,b)$ be the generating function of all the one-particle trajectories passing through the sites a,x,x,b at moments of time $0,t',t'+1,t$, respectively, and making a vertical step with weight y between t' and $t'+1$. Then, the contribution from diagram (i) can be written in the form

$$W_1 = W(x_1^0,x|z|x+1,x_1)W(x_2^0,x+1|y|x+1,x_2). \tag{5}$$

The contribution from diagram (ii) is

$$W_2 = -W(x_1^0,x|z|x,x_1)W(x_2^0,x+1|y|x+1,x_2). \tag{6}$$

Consider now the trajectories with interchanged endpoints [see Fig. 1(b)]. The contribution from these diagrams is

$$W(x_1^0,x|z|x+1,x_2)W(x_2^0,x+1|y|x+1,x_1) - W(x_1^0,x|z|x,x_2)W(x_2^0,x+1|y|x+1,x_1).$$

We are going to take the diagrams in Fig. 1(b) with opposite signs to cancel W_1+W_2. The left-hand side diagrams in Figs. 1(a) and 1(b) are equivalent, however the right ones are different. To cancel all unwanted diagrams, we add to the diagrams in Fig. 1(b) a set of auxiliary trajectories. Namely, we add to the trajectories of the second particle those starting from site x_2^0-1 taken with a minus sign. Also, we add to the trajectories of the first particle a set of trajectories starting from the sites shifted in the negative direction of the infinite chain: x_1^0-1, x_1^0-2, x_1^0-3, …. Then, the contribution from diagram (i) in Fig. 1(b) will be

$$\widetilde{W}_1 = W_1^+ W_1^-, \tag{7}$$

where

$$W_1^+ = \sum_{k=0}^{\infty} W(x_1^0-k,x-k|z|x+1-k,x_2) \tag{8}$$

and

$$W_1^- = W(x_2^0,x+1|y|x+1,x_1) - W(x_2^0-1,x|y|x,x_1). \tag{9}$$

Correspondingly, for the contribution of diagram (ii) in Fig. 1(b) we have

$$\widetilde{W}_2 = W_2^+ W_2^-, \tag{10}$$

where

$$W_2^+ = -\sum_{k=0}^{\infty} W(x_1^0-k,x-k|z|x-k,x_2) \tag{11}$$

and $W_2^- = W_1^-$. Taking into account that the generating functions of trajectories from $(x_i^0-k,0)$, $i=1,2$, to $(x-k,t')$ are equal for all k due to translation invariance, one can check the identity

$$W_1 + W_2 - \widetilde{W}_1 - \widetilde{W}_2 = 0, \tag{12}$$

by comparing all positive and negative terms.

Consider now the evaluation of the two-particle equal-time conditional probability $P_2(x_1,x_2;t|x_1^0,x_2^0;0)$. In this case, we have

$$\begin{aligned} P_2 &= B_0(x_1-x_1^0,t)B_0(x_2-x_2^0,t) - [B_0(x_1-x_2^0,t) \\ &\quad - B_0(x_1-x_2^0+1,t)]\sum_{k=0}^{\infty} B_0(x_2-x_1^0+k,t) \\ &= B_0(x_1-x_1^0,t)B_0(x_2-x_2^0,t) - B_{-1}(x_1-x_2^0,t)B_1(x_2-x_1^0,t). \end{aligned} \tag{13}$$

Indeed, the first term in Eq. (13) generates all possible free trajectories from the starting to the ending space-time point. When one particle approaches another, the second term produces trajectories canceling the unwanted terms. On the other hand, the order of the starting and ending sites in the second term is interchanged. Therefore, each trajectory from the second term, starting from the sites x_1^0-k, $k=0,1,2,\ldots$, or x_2^0-l, $l=0,1$, approaches at least once the space-time point (x,t') or $(x+1,t')$, where it participates in the cancellation procedure.

Next, each free trajectory from site a to site b which makes the vertical step at site x can be decomposed into two parts: $W(a,x|1|x,b)+W(a,x|-z|x,b)$. The second part is unwanted and cancelled, but the first one corresponds to trajectories which continue with the correct weights up to the next collision. As the second term in Eq. (13) contains intersecting trajectories only, all of them cancel out eventually, and only the allowed trajectories from the first term survive.

Let us consider now the case when the number of particles $n\geqslant 3$. First, note that any two intersecting trajectories are nonequivalent: one of them belongs to the overtaking particle and we call it "active," while the trajectory of the overtaken particle we call "passive." In the case of an infinite lattice, the active and passive trajectories are ordered: for each pair of consecutive labels $i,i+1$, the trajectory of the ith particle with respect to the trajectory of the $(i+1)$-st particle on its right-hand side is always active.

Assume that the trajectory of a given particle has m active intersections. It means that it participates m times in the cancellation procedure and its starting point is shifted m times to an arbitrary number of lattice spacings in the negative direction of the chain. As a result, the auxiliary set associated with the free trajectory between sites x_i^0 and x_j becomes

$$B_0(x_j-x_i^0,t) \to \sum_{k=0}^{\infty} \binom{k+m-1}{m-1} B_0(x_j-x_i^0+k,t), \tag{14}$$

because the shift by k positions for m attempts can be done in $(k+m-1)!/(m-1)!k!$ ways. The above result can be expressed in the operator form

$$B_0(x_j-x_i^0,t) \to \frac{1}{(1-\hat{a}_i)^m} B_0(x_j-x_i^0,t), \tag{15}$$

where the operator $\hat{a}_i$ shifts x_i^0 by one step in the negative direction. Similarly, for the trajectories with m passive intersections we obtain

$$B_0(x_j-x_i^0,t) \to \sum_{k=0}^{m} (-1)^k \binom{m}{k} B_0(x_j-x_i^0+k,t), \tag{16}$$

because the right-hand side results from the action of the operator $(1-\hat{a}_i)^m$.

When the number of particles $n\geqslant 3$, the elementary squares shown in Fig. 1 may occur several times in one horizontal strip of Λ. If the squares filled by interacting trajectories are separated from one another by a gap of empty sites, the above arguments can be applied to each pair of interacting trajectories separately. The crucial case for the Bethe ansatz is a situation in which the elementary squares are nearest neighbors. The specific property of the TASEP with backward-ordered update is that, in each pair of interacting trajectories, the right trajectory remains free and interacts with the next trajectory on the right, independently of its left neighbors. Therefore, we can analyze the interaction between particles by successively considering the adjacent elementary squares in each row from left to right, starting from an arbitrary empty square, and then from top to bottom of the lattice, until all unwanted trajectories on Λ are removed.

Thus, we obtain the following discrete-time generalization of the determinant formula derived by Schütz [23]:

Theorem 1. Let the stochastic particle dynamics be given by the TASEP with discrete-time backward-ordered update. Then, the conditional probability $P_n(x_1,\ldots,x_n,t|x_1^0,\ldots,x_n^0;0)$ of finding n particles on the ordered set of sites $x_1<x_2<\ldots<x_n$ at time t, provided at the initial moment of time they have occupied the set of sites $x_1^0<x_2^0<\ldots<x_n^0$, is given by the determinant of the $n\times n$ matrix M:

$$P_n=\det M, \quad \text{with} \quad M_{ij}=B_{i-j}(x_i-x_j^0;t). \tag{17}$$

The continuous-time result [23] follows from this expression in the limit (2), which amounts to the substitution of the Bernoulli distribution by the corresponding Poisson distribution.

III. GENERALIZED UNEQUAL-TIME PROBABILITY

In the general case, particle trajectories on the lattice Λ start from a set of different space-time points, $(x_1^0,t_1^0),\ldots,(x_n^0,t_n^0)$, and end up on a set of different space-time points, $(x_1,t_1),\ldots,(x_n,t_n)$. The generalized conditional probability $P^{(n)}(x_1,t_1;\ldots;x_n,t_n|x_1^0,t_1^0;\ldots;x_n^0,t_n^0)$ is given by the weighted sum of all such trajectories which obey the exclusion rule (a) and have weights corrected according to rule (b) (see Sec. II).

Let us turn again to the cancellation procedure described in the previous section. The conditions under which the above procedure works can be formulated as follows:

(1) The united trajectory set of every pair of particles contains a subset of nonintersecting trajectories.

(2) The interacting trajectories produce only two types of unwanted diagrams shown in Fig. 1(a).

(3) Under permutation of the endpoints of two trajectories, one obtains crossing trajectories which produce only two types of unwanted diagrams shown in Fig. 1(b).

We shall find restrictions on the positions of the different starting and ending space-time points of the particles under which the above conditions hold true.

To this end, we give some definitions which make conditions (1)–(3) more transparent and convenient to work with.

For each particle, $i=1,2,\ldots,n$, all the possible free trajectories with specified space-time endpoints, $v_i^0:=(x_i^0,t_i^0)\in\Lambda$ and $v_i:=(x_i,t_i)\in\Lambda$, are confined to a parallelogram composed of vertical and diagonal lattice bonds only, whose uppermost corner is v_i^0, and the lowermost one is v_i. This parallelogram will be called *trajectory set* of particle i and will be denoted by $\mathcal{T}(v_i^0,v_i)$, or in short notation, $\mathcal{T}_i$.

Two parallelograms $\mathcal{T}_i$ and $\mathcal{T}_j$ *intersect* if they have a common space-time point.

Two parallelograms $\mathcal{T}_i$ and $\mathcal{T}_j$ *interact* if they intersect or if there is a pair of vertical bonds $b_i\in\mathcal{T}_i$ and $b_j\in\mathcal{T}_j$ belonging to the same elementary square of the lattice Λ.

The interaction between two parallelograms is called *fatal interaction* if each trajectory from one parallelogram intersects all the trajectories from the other.

We call the *cone of a starting point* (x_i^0,t_i^0) the set of space-time points (x,t) such that a free-particle trajectory starting from (x_i^0,t_i^0) can reach (x,t):

$$B_0(x-x_i^0,t-t_i^0)\neq 0. \tag{18}$$

The *cone of an ending point* (x_i,t_i) is defined as the set of space-time points (x,t) such that a free-particle trajectory starting from (x,t) can reach (x_i,t_i):

$$B_0(x_i-x,t_i-t)\neq 0. \tag{19}$$

Consider now a pair of *interacting* trajectory sets $\mathcal{T}_i$ and $\mathcal{T}_j$. We say that two trajectory sets interact *invasively* if the starting or ending point of one of the sets belongs to the cone of the starting or ending point, respectively, of the other set.

Finally, we define the *connectedness* property of a finite collection of trajectory sets in terms of their pairwise interaction. Any connected collection of $k\geqslant 2$ interacting trajectory sets will be called a *connected cluster* of trajectories. Generally, the connectedness relation splits the collection $\{\mathcal{T}_i, i=1,\ldots,n\}$ of trajectory sets into noninteracting among themselves (independent) components. Since the conditional probability $P^{(n)}(x_1,t_1;\ldots;x_n,t_n|x_1^0,t_1^0;\ldots;x_n^0,t_n^0)$ factorizes into a product of conditional probabilities describing each independent component separately, it suffices to obtain the expression for just one connected cluster of trajectory sets.

Let us specify also the particle labeling rule. In each connected cluster of n particles, we label the particles in ascending order according to the rule $i<j$, where $i,j\in\{1,2,\ldots,n\}$, if $x_i^0<x_j^0$, or if $x_i^0=x_j^0$, then $t_i^0<t_j^0$.

Now we can formulate our main theorem:

Theorem 2. Let the stochastic particle dynamics be given by the TASEP with discrete-time backward-ordered update. Let, in addition, the trajectory sets $\mathcal{T}_i$, $i=1,2,\ldots,n$, of the particles belong to a single connected cluster and obey the conditions for nonfatal and noninvasive interaction between any pair of trajectory sets $\mathcal{T}_i$ and $\mathcal{T}_j$, $1\leqslant i,j\leqslant n$.

Then, the conditional probability

$$P^{(n)}(x_1,t_1;\ldots;x_n,t_n|x_1^0,t_1^0;\ldots;x_n^0,t_n^0)$$

is given by the determinant of the $n\times n$ matrix $M^{(n)}$,

$$P^{(n)}=\det M^{(n)}, \quad \text{with} \quad M_{ij}^{(n)}=B_{i-j}(x_i-x_j^0,t_i-t_j^0). \tag{20}$$

Proof. First, we show that the conditions of the theorem ensure the fulfillment of conditions (1)–(3):

(1) The condition for nonfatal interaction between any pair of trajectory sets is equivalent to condition (1).

(2) The condition for noninvasive interaction excludes the possibility of the formation of unwanted terms of other types, different from those described in Sec. II. To clarify this point, let us turn to Fig. 2.

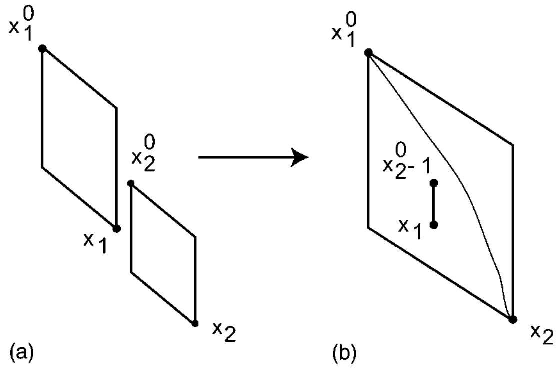

FIG. 2. An example of invasively interacting trajectory sets (a) and corresponding auxiliary trajectory sets (b).

An example of two invasively interacting trajectory sets is shown in Fig. 2(a). Trajectory sets corresponding to the term $B_0(x_2-x_1^0,t_2-t_1^0)B_0(x_1-x_2^0+1,t_1-t_2^0)$, generated by the cancellation procedure, are shown in Fig. 2(b). These sets describe the auxiliary trajectories constructed under permutation of the space-time ending points of the particles and shifted by one site to the left starting point of the right-hand side particle (at that its trajectory set degenerates to a vertical segment). As it is seen from the figure, in this case there exist noncrossing particle trajectories which do not match any unwanted terms subject to cancellation. Therefore, the invasive interaction violates condition (3) for applicability of the cancellation procedure. Note that if a starting or ending point of one of the particles belongs to the trajectory set of another, condition (2) will be violated as well. As it is readily seen, conditions (2) and (3) hold true if the starting point of the right-hand side particle does not belong to the cone of the starting point of the left-hand side particle and, similarly, the ending point of the left-hand side particle does not belong to the cone of the ending point of the right-hand side one. In summary, the condition for noninvasive interaction implies the validity of conditions (2) and (3).

Consider now the case of two particles for which the conditions of the theorem are fulfilled. Assume that their first nearest neighborhood occurs at x and $x+1$ at time t. There exist unwanted terms, similar to those considered in Sec. II [see Fig. 1(a)], with the difference that the starting and ending space-time points may have unequal times,

$$W(x_1^0,t_1^0;x,t|z|x+1,t+1;x_2,t_2) \times W(x_2^0,t_2^0;x+1,t|y|x+1,t+1;x_1,t_1), \tag{21}$$

$$W(x_1^0,t_1^0;x,t|-z|x,t+1;x_2,t_2) \times W(x_2^0,t_2^0;x+1,t|y|x+1,t+1;x_1,t_1). \tag{22}$$

The derivation of the terms canceling (21) and (22) is analogous to one given in Sec. II:

$$\widetilde{W}_1 = W_1^+ W_1^-, \tag{23}$$

$$\widetilde{W}_2 = W_2^+ W_2^-, \tag{24}$$

where

$$W_1^+ = \sum_{k=0}^{\infty} W(x_1^0-k,t_1^0;x-k,t|z|x+1-k,t+1;x_2,t_2), \tag{25}$$

$$W_2^+ = -\sum_{k=0}^{\infty} W(x_2^0-k,t_2^0;x-k,t|z|x-k,t+1;x_1,t_1), \tag{26}$$

and

$$W_1^- = W_2^- = W(x_2^0,t_2^0;x+1,t|y|x+1,t+1;x_1,t_1) - W(x_2^0-1,t_2^0;x,t|y|x,t+1;x_1,t_1). \tag{27}$$

Let us verify the identity $W_1+W_2=\widetilde{W}_1+\widetilde{W}_2$:

$$\begin{aligned}
&\sum_{k=0}^{\infty}[W(x_1^0-k,t_1^0;x-k,t|z|x+1-k,t+1;x_2,t_2)\\
&\quad - W(x_1^0-k,t_1^0;x-k,t|z|x-k,t+1;x_2,t_2)]\\
&\quad \times W^-(x_2^0,t_2^0;x+1,t|y|x+1,t+1;x_1,t_1)\\
&= \{W(x_1^0,t_1^0;x,t|z|x+1,t+1;x_2,t_2)\\
&\quad + \sum_{k=0}^{\infty}[W(x_1^0-k-1,t_1^0;x-k-1,t;|z|x-k,t+1;x_2,t_2)\\
&\quad - W(x_1^0-k,t_1^0;x-k,t|z|x-k,t+1;x_2,t_2)]\}\\
&\quad \times W^-(x_2^0,t_2^0;x+1,t|y|x+1,t+1;x_1,t_1)\\
&= W(x_1^0,t_1^0;x,t|z|x+1,t+1;x_2,t_2)\\
&\quad \times W^-(x_2^0,t_2^0;x+1,t|y|x+1,t+1;x_1,t_1).
\end{aligned} \tag{28}$$

These are exactly the unwanted terms represented graphically in Fig. 1(b). The expression

$$W(x_1^0,t_1^0;x,t|z|x+1,t+1;x_2,t_2) \times W(x_2^0,t_2^0;x+1,t|y|x+1,t+1;x_1,t_1) \tag{29}$$

corresponds to case (i), and the expression

$$-W(x_1^0,t_1^0;x,t|z|x+1,t+1;x_2,t_2) \times W(x_2^0-1,t_2^0;x,t|y|x,t+1;x_1,t_1) \tag{30}$$

to case (ii).

Consider next a moment of time $t'>t$, when not only free trajectories come to the neighboring sites x' and $x'+1$, but trajectories which have already interacted at time t. Let us write down the corresponding terms separately:

$$\begin{aligned}
&W^+(x_1^0,t_1^0;x,t|1|x+1,t+1,x',t'|z|x'+1,t'+1;x_2,t_2)\\
&\quad \times W^-(x_2^0,t_2^0;x+1,t|y|x+1,t+1;x'+1,t'|y|x'+1,t'\\
&\quad +1;x_1,t_1),
\end{aligned} \tag{31}$$

$$W^{+}(x_1^0,t_1^0;x,t|1|x+1,t+1,x',t'|-z|x',t'+1;x_2,t_2)$$
$$\times W^{-}(x_2^0,t_2^0;x+1,t|y|x+1,t+1;x'+1,t'|y|x'+1,t'$$
$$+1;x_1,t_1), \quad (32)$$

$$\overline{W}^{+}(x_1^0,t_1^0;x',t'|z|x'+1,t'+1;x_2,t_2)$$
$$\times W^{-}(x_2^0,t_2^0;x'+1,t'|y|x'+1,t'+1;x_1,t_1), \quad (33)$$

$$\overline{W}^{+}(x_1^0,t_1^0;x',t'|-z|x',t'+1;x_2,t_2)$$
$$\times W^{-}(x_2^0,t_2^0;x'+1,t'|y|x'+1,t'+1;x_1,t_1). \quad (34)$$

The first two terms (31) and (32) describe trajectories which have already interacted at time t, and the next two terms (33) and (34) describe trajectories which have not interacted by time t'. To distinguish the latter, the symbols of their generating functions are marked with a bar.

By adding up (31) and (32), as well as (33) and (34), we obtain the necessary corrections to the free trajectories up to time t', when they come to the neighboring sites x' and $x'+1$,

$$W(x_1^0,t_1^0;x,t|1|x+1,t+1;x',t'|z|x'+1,t'+1;x_2,t_2)$$
$$\times W^{-}(x_2^0,t_2^0;x+1,t|y|x+1,t+1,x'+1,t'|y|x'+1,t'$$
$$+1;x_1,t_1)+\overline{W}(x_1^0,t_1^0;x',t'|z|x'+1,t'+1;x_2,t_2)$$
$$\times W^{-}(x_2^0,t_2^0;x'+1,t'|y|x'+1,t'+1;x_1,t_1). \quad (35)$$

Next, by moving down with time and successively considering the adjacent elementary squares in each row, we repeat the analysis of the two-particle interaction until all the trajectories are considered: those which have interacted earlier in time, as well as the free noninteracting ones.

To demonstrate that all the unwanted contributions from the free-particle trajectories are canceled out by the terms entering into the expression $B_1(x_2-x_1^0;t_2-t_1^0)B_{-1}(x_1-x_2^0;t_1-t_2^0)$, we sum up over all the auxiliary trajectories that participate in the cancellation. First, we note that, as in the equal-time case considered in Sec. II, in each pair of interacting particles, the particle on the right-hand side remains free (and eventually interacts with the particles on the right, independently of its left neighbor). Let the trajectory of the second particle (with $x_2^0>x_1^0$) be such that the two particles become nearest neighbors at the following set of space-time points:

$$\{(x,t),(x+1,t)\};\{(x',t'),(x'+1,t')\};\ \dots\ ;$$
$$\{(x^{(n)},t^{(n)}),(x^{(n)}+1,t^{(n)})\}.$$

The corresponding contribution from the auxiliary trajectories of the second particle is

$$W^{-}(x_2^0,t_2^0;x+1,t|y|x+1,t+1;$$
$$x'+1,t'|y|\dots|y|x^{(n)}+1,t^{(n)}+1;x_1,t_1).$$

This expression can be taken out as a common factor. The auxiliary trajectories of the first particle contribute the following terms:

$$W^{+}(x_1^0,t_1^0;x,t|z|x+1,t+1;x_2,t_2)$$
$$+W^{+}(x_1^0,t_1^0;x,t|-z|x,t+1;x_2,t_2)$$
$$+W^{+}(x_1^0,t_1^0;x,t|1|x+1,t+1;x',t'|z|x'+1,t'+1;x_2,t_2)$$
$$+W^{+}(x_1^0,t_1^0;x,t|1|x+1,t+1;x',t'|-z|x'+1,t'+1;x_2,t_2)$$
$$+\overline{W}^{+}(x_1^0,t_1^0;x',t|z|x'+1,t+1;x_2,t_2)$$
$$+\overline{W}^{+}(x_1^0,t_1^0;x',t|-z|x',t+1;x_2,t_2)+\dots \quad (36)$$

Adding up all these terms we obtain $B_1(x_2-x_1^0;t_2-t_1^0)$. Therefore, the contribution of the considered pairs of auxiliary trajectories to the cancellation is given by the expression

$$B_1(x_2-x_1^0;t_2-t_1^0)W^{-}(x_2^0,t_2^0;x+1,t|y|x+1,t+1;$$
$$x'+1,t'|y|\dots|y|x^{(n)}+1,t^{(n)}+1;x_1,t_1).$$

Summing up over all the auxiliary trajectories of the second particle, we obtain the result $B_1(x_2-x_1^0;t_2-t_1^0)B_{-1}(x_1-x_2^0;t_1-t_2^0)$, which proves the determinant formula (20) for the case of two particles.

Passing to the consideration of a connected cluster of $n\geqslant 3$ particles, we note that the case $x_i^0=x_j^0$, $i\neq j$, is excluded by the condition for noninvasive interaction. Therefore, particles may not start from the same site at different times and according to our particle labeling rules, $x_1^0<x_2^0<\cdots<x_n^0$. Simple geometrical considerations show that a particle cannot interact nonfatally and noninvasively with two or more right (or left) neighbors existing in different time intervals. Thus we can begin with the first (leftmost) particle and consider its interaction with the second one (its right neighbor). As in the case of equal times of start and finish, in each pair of interacting trajectories the right one remains free and interacts with the trajectory on its own right-hand side independently of the left neighbors. The trajectory of the nth, rightmost particle is free. Therefore, we can carry out the above analysis by successively considering all the pairs of interacting particles in the connected cluster until all unwanted trajectories are removed.

IV. ZERO-RANGE PROCESS

In this section we consider a zero-range discrete-time process on a finite chain of L sites with integer coordinates $i=1,\dots,L$ and open boundary conditions. The configuration of the system is specified by the occupation numbers $n_i(t)$, $i=1,\dots,L$, at discrete moments of time $t=0,1,\dots$. The probabilistic dynamics of the system is given by the probability of particle hopping from site i to the nearest-neighbor site on the right $i+1$. We assume that the hopping probabilities are independent of i and t, but depend on the order in which particles have arrived at the site i. Namely, out of all $n_i(t)$ particles on site i, the particle that has arrived first leaves that site with probability z. The remaining $n_i(t)-1$ particles remain on site i at the moment t. The hopping process conserves the total number of particles n. We assume that particles are injected on the first site $i=1$ of the chain at given moments of time $t_1^0,t_2^0,\dots,t_n^0$ and leave the system from the last site $i=L$ at given moments of time $t_1,t_2,\dots,t_n$.

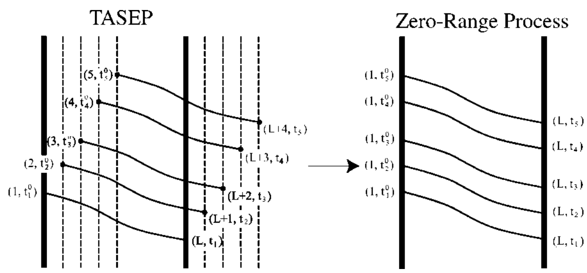

FIG. 3. Correspondence between TASEP and zero-range process.

The probability of this event is denoted by $P_L(t_1^0,t_2^0,\ldots,t_n^0|t_1,t_2,\ldots,t_n)$.

Theorem 3. Given the sets of times $\{t_i^0\}$ and $\{t_i\}$, $i=1,2,\ldots,n$, the conditional probability $P_L(t_1^0,t_2^0,\ldots,t_n^0|t_1,t_2,\ldots,t_n)$ is given by the determinant of the $n\times n$ matrix $M^{(n)}$,

$$P_L = \det M^{(n)}, \quad \text{where} \quad M_{ij}^{(n)} = B_{i-j}(L-1+i-j,t_i-t_j^0). \tag{37}$$

Proof. Consider the TASEP for n particles starting at space-time points (x_i^0,t_i^0) and ending at (x_i,t_i), $i=1,2,\ldots,n$ where $x_i^0=i$ and $x_i=L+i-1$. Shifting the ith trajectory as a whole by $i-1$ sites to the left, we obtain the ZRP shown in Fig. 3, which is the object of Theorem 3. Then, Theorem 2 proves Eq. (37).

In the continuous-time limit we obtain a standard ZRP with constant hopping rates which are independent of the occupation number of the sites.

V. DISCUSSION

Determinant expressions enumerating configurations of nonintersecting trajectories have appeared in physical literature in the early sixties, in the context of exactly solvable lattice models of statistical mechanics such as the dimer model on the hexagonal lattice [32], the models of two-dimensional biomembrane [33], and the free-fermion sector of the six-vertex model [34,35]. In 1984, Fisher considered this problem in a frame of the random walk theory and introduced vicious walkers to coin the condition of nonmeeting of different particles. This line was continued later by Forrester [36].

Independently, the problem of mutually avoiding trajectories was considered in combinatorics, where the determinant expression for the number of configurations is known as the Gessel-Viennot theorem [37]. Recently, a connection between the statistical mechanics approach and the Gessel-Viennot theorem has been established [38,39], and more general cases of interaction between walkers have been considered [40].

The difference between vicious walkers and the ASEP is in the statistical weight of bundles of trajectories. A vicious walker does not see neighboring walkers until it collides with them. Therefore, the weights of parts of its trajectory do not depend on neighboring trajectories during the whole survival time interval. On the contrary, in the ASEP, the probability of a step depends on a state of target site: if it is occupied, the step is forbidden with probability one. This difference puts two kinds of models into different classes. If the vicious walkers belong to the class of free-fermion models, the ASEP is the model of essentially interacting particles. Using terminology of the Bethe ansatz, the free-fermion models are solved by purely antisymmetric Bethe functions, whereas the ASEP needs less trivial Bethe ansatz. Nevertheless, it has been shown by Schütz that the specific form of the ASEP interaction still allows a determinant representation, although the simple binomial matrix elements in the Gessel-Viennot determinant should be replaced by more complicated functions (actually, by infinite sums of binomial coefficients). An analytical derivation of the determinant formula for the ASEP on an infinite lattice is given in [23]. In this paper, we give a geometrical interpretation of this solution using a trajectory analysis which has been presented first in [31] as a part of solution of the ASEP on a ring.

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